I. INTRODUCTION

At any flow rate, the flow through a helical coil is uniquely different from that through a straight pipe due to the secondary motion induced by an imbalance between the cross-stream pressure gradient and the centrifugal force. Figure 1 is a cartoon of the resulting fully developed secondary flow streamlines. The two counter-rotating vortices, called Dean vortices, are present, even for the most mildly curved pipe. At low Reynolds number, the fully developed Dean vortex pattern is steady and laminar. With increasing Reynolds number, previous experimental measurements at a fixed location in the coil have revealed a nonturbulent flow unsteadiness due to a traveling wave instability. The objective of this investigation is to numerically calculate the experimentally observed Dean vortex flow and its associated traveling wave. Flow visualization photographs are also presented for the entire laminar/turbulent flow transitional regime.

Figure 1 shows the toroidal coordinate system used to describe the curved pipe geometry. The three orthogonal coordinate directions are \( r \), the radial distance from the pipe center; \( \phi \), the poloidal angle in the pipe cross section; and \( \theta \), the toroidal angle. The radius of the curvature to pipe radius ratio is denoted by \( R_c/a \). The Reynolds number is defined as \( Re=Ud/\nu \), where \( d (=2a) \) is the pipe diameter, \( U \) is the average streamwise velocity, and \( \nu \) is the fluid kinematic viscosity. The Dean number is defined as \( De=Re(a/R_c)^{1/2} \).

At least three experimental studies have noted the presence of a traveling wave instability in a helically coiled pipe. In his visualization experiments for coils with \( R_c/a = 18.7 \) and 31.9, Taylor\(^1\) observed that as \( Re \) was increased a dye streak introduced into the flow began to oscillate at \( Re = 5830 \) and 5010, respectively. However, the undulating dye streak remained coherent, indicating that the unsteadiness was not due to turbulent fluctuations. Increasing the Reynolds number beyond \( Re = 7100 \) and 6350, respectively, produced turbulent fluctuations in the flow.

Sreenivasan and Strykowski\(^2\) plotted velocity time records measured with hot wire anemometers placed along the midplane of a coiled pipe with \( R_c/a = 17.2 \), at distances one-quarter of the radius from the inner and outer walls. The velocity time records measured near the inner wall showed periodicity at a single frequency for \( Re > 5000 \). A second harmonic of the main frequency was observed as \( Re \) increased. Low-frequency unsteadiness was not observed near the outer wall. These measurements indicate that the traveling wave instability is localized in the inner half of the pipe cross section.

Webster and Humphrey\(^3\) present midplane profiles of the mean and rms values of two velocity components, measured by means of laser Doppler velocimetry (LDV) for the fully developed flow through a helically coiled pipe with \( R_c/a = 18.2 \). In the range \( 5060 < Re < 6330 \) (1190 < \( De < 1480 \), the velocity component time records revealed periodic oscillations of nondimensional frequencies, \( fd/U = 0.25 \) and 0.5 in the inner half of the pipe cross section due to a traveling wave instability. Low-frequency unsteadiness was not observed near the outer wall. A local increase in the rms velocity profile at the inner side of the pipe center due to the traveling wave instability was observed for \( Re > 5060 \).

These experimental observations conclusively show that, in a range of Reynolds number, a traveling wave instability exists in a helically coiled pipe. However, the cause of the instability remains unknown. In this investigation the unsteady three-dimensional flow through a helically coiled pipe is calculated numerically and, for the conditions explored, a...
deterministic traveling wave perturbation to the fully developed Dean vortex flow solution is found. Disorganized or chaotic behavior is not observed numerically. The radius of curvature ratio for the calculation is equal to that of the experimental apparatus of Webster and Humphrey,3 $R_c/a = 18.2$. Calculations have been performed for one Reynolds number only, in the range for which the traveling wave was experimentally observed; $Re = 5480$ ($De = 1280$). The objective of the calculations is to reveal the physics of the traveling wave instability.

II. FLOW VISUALIZATION

There are three fundamental flow regimes for a helical coil and its upstream straight pipe tangent. For $Re < 2300$, approximately, the flow in the upstream tangent is laminar and the flow in the coil is laminar. For $Re > 8000$ the flow in the upstream tangent is turbulent and the flow in the coil is also turbulent. (Note that the Re for completely turbulent flow in a coil is a function of $R_c/a$, and the value of 8000 corresponds to $R_c/a = 18.2$.) In the range $2300 < Re < 8000$, the flow in the upstream tangent is turbulent but that in the coil is laminarized. In this intermediate range of Reynolds number, turbulent fluctuations present in the inlet flow are damped either completely or quite significantly in the coil, depending on the Reynolds number. Obviously, the expression “transition to turbulence” as it applies to straight pipes is inappropriate or, at best, ambiguous when applied to the flow through a helical coil. Nevertheless, for ease of discussion the regime spanning steady laminar flow and fully turbulent flow in a coil ($2300 < Re < 8000$) is referred to here as the “transitional” regime. The objective of the flow visualization is to demonstrate the laminarized flow and, to the extent possible, characterize the observed traveling wave instability through estimates of its wavelength and wave speed.

The flow visualization was performed by injecting a methylene blue dye streak in the fully developed flow region of the coil. The apparatus has been previously described by Webster and Humphrey and consists of a closed loop, constant pressure head water flow system. The test section is a helically coiled Tygon tube with inner diameter equal to 3.81 ± 0.05 cm. A hypodermic needle penetrated the wall of the tube at the fifth coil turn. The needle was nominally located in the midplane of the cross section at the inner curvature wall. The needle passed perpendicularly through the tube wall and then made a 90° turn so that its tip faced downstream. The precise location was difficult to control and the needle tip appeared to be slightly above the geometric midplane. The needle was close to the tube wall (within 0.2 cm) and did not appear to perturb the flow significantly. The methylene blue dye was sufficiently diluted in water to be essentially neutrally buoyant. The dye reservoir was located at a height above the constant head tank and was connected to the needle with 0.3175 cm inner diameter Tygon tubing. The dye flow rate was controlled with a constriction clamp so that the speed of the colored water entering the coil essentially matched that of the streamwise flow at the injection point in the coil.

The coiled Tygon tube wall was sufficiently clear to allow optical access for photographic purposes. A scale 50° long and of 1° increments was drawn on the tube wall. The flow was illuminated from the center of the coil using two lamps with blue 100 W light bulbs. The still photographs of the dye streak were taken using a Canon T90 camera and 200 ISO Kodak color print film. The video recording was made using a Panasonic WV-2170 video camera, a Panasonic AG-6300 tape drive, a 20 in. Sony Trinitron monitor, and a TDK VHS videotape. The camera was located at a distance of approximately 0.3 m from the coiled tube wall and was focused on the midpoint of the drawn scale.

Similar locations within a wave cycle (i.e., a maximum or a minimum) were found and the streamwise arclengths between these pairs of points were determined by using the scale marked on the tube wall and shown in the photographs. For each $Re$, the results from 20 photographs were averaged. The video recordings were used to estimate the wave speed based on the time required to displace the wave a measured distance. The time period was determined from the on-screen chronometer (±0.01 s), and the results of 20 estimations of the wave speed were averaged. The uncertainty estimates of the wavelength measured from the still photographs and of the wave speed measured from the video recordings are both ±5% of the measured value due to the uncertainty of the scale, the wave position, and the relative camera angle. The uncertainty in the Reynolds number was ±4%.

In all of the photographs shown in Fig. 2 the flow is from right to left. Figure 2(a) shows a photograph of the dye streak at $Re = 3800$ ($De = 890$). The dye streak is smooth and shows neither a perturbation due the traveling wave nor diffusive mixing due to small-scale turbulent velocity fluctuations. This is consistent with the LDV measurements of Webster and Humphrey, which indicate that the traveling wave instability is first observed around $Re = 5060$ ($De = 1190$) and that turbulent fluctuations are suppressed until approximately $Re = 6330$ ($De = 1480$). The photograph in Fig. 2(b) shows the dye streak at $Re = 5060$ ($De = 1190$). A periodic perturbation of the dye streak with a wavelength of approximately 18.4° is observed at this $Re$. The dye streak does not cross the geometric midplane, thus suggesting that the instability is a varicose mode. Figures 2(c), 2(d), and 2(e) are the photographs for $Re = 5480$, 5900, and 6330 ($De = 1280$, 1380, and 1480), respectively. A perturbation of the dye streak is observed in each photograph and the disorganized nature of the perturbation increases with increasing $Re$.  

FIG. 1. Definition of the toroidal coordinate system with a cartoon of the cross-stream flow streamlines.
The averaged measurements of wavelength and wave speed are given in Table I for Re=5060 and 5480. For Re=5900 and 6330 the disorganized nature of the perturbed dye streak prevents accurate estimates of these quantities. The wavelength ($\lambda$) estimates from the still photographs and the wave speed ($c$) estimates from the video recording can be used to calculate the wave frequency ($f$) from the following relationship:

$$f = c/\lambda.$$  

The measured wave frequency is also known from the LDV time records of Webster and Humphrey\(^3\) and the average value of the first harmonic is tabulated in Table I together with that obtained from Eq. (1). A comparison of these values of $f$ demonstrates that they agree to within the measurement uncertainty. Knowing the wavelength is crucial for the computational effort since the boundary conditions in the streamwise direction need to be appropriately applied. As discussed below, the length of the computational domain is made equal to the measured wavelength and the boundary condition in the streamwise direction is specified as periodic.

### III. CONSERVATION EQUATIONS

The equations describing the constant property flow through a coiled pipe are the continuity and Navier–Stokes equations for the conservation of mass and momentum. The finite pitch angle of the experimental apparatus of Webster and Humphrey\(^3\) is small (3.4°) and, from the work of Murakami et al.\(^4\) and Liu et al.,\(^5\) is expected to have a negligible effect on the flow. Thus, the pitch angle has been neglected in deriving the conservation equations in toroidal coordinates. The conservation equations in dimensional component form are as follows: Continuity,

$$\frac{1}{r \xi} \left( \frac{\partial}{\partial r} (ru_r) + \frac{\partial}{\partial \phi} (ru_\phi) + \frac{\partial}{\partial \theta} (ru_\theta) \right) = 0;$$

r momentum,

$$\frac{\partial u_r}{\partial t} + (\vec{u} \cdot \nabla) u_r - \frac{u_r^2}{r} - \frac{\cos \phi}{\xi} u_\phi^2 = -\frac{1}{\rho} \frac{\partial p}{\partial r} + \nu \left( \frac{\partial^2 u_r}{\partial r^2} - \frac{2}{r^2} \frac{\partial u_r}{\partial \phi} - \frac{\partial u_r}{\partial \phi} - \frac{2}{r} u_r + \frac{\sin \phi}{r \xi} u_\phi \right) + \cos \phi \frac{1}{\xi^2} \left( u_\phi \sin \phi - u_r \cos \phi - \frac{2}{r} \frac{\partial u_\phi}{\partial \theta} \right);$$

$$\phi$$ momentum,

$$\frac{\partial u_\phi}{\partial t} + (\vec{u} \cdot \nabla) u_\phi + \frac{u_r u_\phi}{r} \frac{\sin \phi}{\xi} u_\phi^2 = -\frac{1}{\rho} \frac{\partial p}{\partial \phi} + \nu \left( \frac{\partial^2 u_\phi}{\partial r^2} + \frac{2}{r^2} \frac{\partial u_\phi}{\partial \phi} - \frac{\partial u_\phi}{\partial \phi} - \frac{2}{r} u_\phi \right) - \frac{\sin \phi}{\xi^2} \left( u_\phi \sin \phi - u_r \cos \phi - \frac{2}{r} \frac{\partial u_\phi}{\partial \theta} \right);$$

At Re=5480 the perturbation still appears to be periodic. However, at Re=5900 and 6330 this is not the case and for Re=6330, especially, the dye streak has diffused due to the emergence of turbulent fluctuations. The gradual appearance of turbulent velocity fluctuations in this range is consistent with the velocity measurements of Webster and Humphrey.\(^3\) At Re=8650 (De=2030) the turbulent velocity fluctuations diffuse the dye to the extent that the streak is indistinguishable in the half-toned image and hence is not shown here.

The averaged measurements of wavelength and wave speed are given in Table I for Re=5060 and 5480. For Re
$\theta$ momentum,

$$
\frac{\partial u_\theta}{\partial t} + (\hat{u} \cdot \nabla) u_\theta + \frac{u_\theta}{\xi} (u_r \, \cos \phi - u_\phi \, \sin \phi) \\
= -\frac{1}{\rho} \frac{\partial p}{\partial \theta} + \frac{2}{\xi^2} \left( \frac{\partial u_r}{\partial \theta} \cos \phi - \frac{\partial u_\phi}{\partial \theta} \right) \\
\times \sin \phi - \frac{u_\theta}{2},
$$

(5)

where $\xi = R_r + r \cos \phi$,

$$(\hat{u} \cdot \nabla) = u_r \frac{\partial}{\partial r} + \frac{u_\theta}{r} \frac{\partial}{\partial \phi} + \frac{u_\phi}{\xi} \frac{\partial}{\partial \theta},$$

and

$$\nabla^2 = \frac{1}{r \xi} \left[ \frac{\partial}{\partial r} \left( r \xi \frac{\partial}{\partial r} \right) + \frac{1}{r} \frac{\partial}{\partial \phi} \left( \xi \frac{\partial}{\partial \phi} \right) + \frac{1}{\xi} \frac{\partial}{\partial \theta} \left( r \frac{\partial}{\partial \theta} \right) \right].$$

The respective curvatures of two of the toroidal coordinate axes ($\phi$ and $\theta$) lead to the appearance of several centrifugal and Coriolis force terms in the equations for conservation of momentum. In the $r$-momentum equation, the two additional terms correspond to centrifugal forces and result from the cross-stream velocity ($u_\phi/r$) and streamwise velocity ($\cos \phi u_\phi/\xi$), respectively. In the $\phi$-momentum equation the terms correspond to the Coriolis force due to the cross-stream velocity ($u_r u_\phi/r$), and the centrifugal force due to the streamwise velocity ($\sin \phi u_\phi/\xi$). In these equations it is the centrifugal force terms involving the streamwise velocity component that are primarily responsible for the counter-rotating Dean vortices shown in Fig. 1. Although the terms comprised of the cross-stream velocity components are relatively small in magnitude, the $u_\phi/r$ term influences the stability of the flow and is discussed further below. The additional terms appearing in the $\theta$-momentum equation correspond to Coriolis forces. Several curvature-related diffusion terms also appear in addition to the Laplacian term ($\nabla^2$) for each momentum equation.

### IV. NUMERICAL CODE

#### A. Numerical method

The CUTFLOWS (Computing Unsteady Three-dimensional Elliptic Flows) numerical algorithm developed by Schuler and Treidler has been modified to compute the present flow. The algorithm is described in Humphrey et al., and its adaptation to the conservation equations in toroidal coordinates is discussed by Webster. The algorithm solves directly for the primitive variables (velocity and pressure) on a staggered grid configuration. The conservation equations are discretized in space by integrating over separate control volumes in each of the three coordinate directions. Central differences are used to represent the diffusion terms and the second-order upstream interpolation method of Leonard is employed for the convective terms. The pressure gradient term and additional convective and diffusive terms that appear due to the coordinate system curvature are integrated over the control volumes and treated as source terms. The resulting set of time-dependent ordinary differential equations for the velocity components at the respective grid nodes is solved by using an explicit second-order accurate Runge–Kutta scheme.

Following the numerical procedure of Chorin, the divergence-free velocity vector is uniquely decomposed into pseudovelocity and pressure gradient components. The pseudovelocity contribution is computed directly using the Runge–Kutta scheme. The pressure gradient contribution is computed by numerically solving the discrete Poisson equation for the pressure that results from the imposition of the divergence-free condition (i.e., the continuity constraint) at the end of each half time step in the Runge–Kutta scheme. The discrete Poisson equation is solved by the conjugate gradient procedure (see Luenberger), the merits of which are discussed by Humphrey et al.

#### B. Calculation domain and boundary conditions

The results reported here correspond to a computational grid with 45 nodes nonuniformly distributed in the radial direction, 33 nodes uniformly distributed in the cross-stream circumferential direction, and 48 nodes uniformly distributed in the streamwise direction: $45 \times 33 \times 48$ (nodes) not including nodes required outside the field to enforce the boundary conditions. The nonuniform spacing of the grid in the radial direction ensured that not less than 12 nodes were distributed across the largest gradients of velocity near the wall (i.e., across a distance approximately 0.1a from the wall). The time step was 0.001 seconds which was small compared to the fundamental time scale of the oscillating flow and ensured that the explicit time integration scheme remained numerically stable.

Preliminary calculations with the entire pipe cross section ($0 < \phi < 2\pi$) revealed that the flow was perfectly symmetric about the midplane (shown by the dotted line on the right-hand side of Fig. 1). Subsequent calculations were performed on the upper half of the pipe cross section ($0 < \phi < \pi$) by imposing the reflection boundary condition at the midplane. The number of nodes in each of the three orthogonal directions was varied in order to determine the required resolution to achieve grid independent results. The mean velocity field varied negligibly among the tested grids. The traveling wave remained qualitatively similar among the tested grids, however, its amplitude varied. Figure 3 shows the amplitude of the traveling wave as a function of the grid refinement, where $|\bar{u}|$ represents the integrated average magnitude of the deviation of the velocity from the mean value over the entire domain volume, $\hat{\theta}$:

$$|\bar{u}| = \frac{1}{\hat{\theta}} \int_0^{\hat{\theta}} |u - \bar{u}| \, d \theta.$$

The mean of a quantity, denoted by an overbar, is defined as the average of that quantity at a point in the pipe cross section over the length of the calculation domain, i.e. one wavelength. Figures 3(a) and 3(b) show that the amplitude of the traveling wave increases with grid refinement in the $\theta$ and $\phi$ directions and the refinement procedure was stopped when additional refinement ceased to change the results appreciably. The grid in the radial direction was sufficiently refined.
to capture the strong gradient of the mean velocity near the pipe wall, hence varying the grid had a negligible effect of the traveling wave amplitude, as shown in Fig. 3(c).

The boundary conditions were imposed as follows. The flow was driven by a specified constant bulk streamwise pressure gradient. Each of the velocity components was set to zero at the wall in order to meet the no-slip, impermeable wall conditions. Following the procedure of Goering and Humphrey and Lavine, the three velocity components at the center node ($r = 0$) were calculated by solving the equations that result from taking the analytical limit of the momentum equations as $r$ approaches zero. The form of the solution to these equations can be found by inspection and the leading coefficients are determined by the velocity values at neighboring nodes. Therefore, at the conclusion of each explicit time step, the center node velocity components were determined from the velocity components at the surrounding grid nodes in each streamwise grid plane. This pseudosteady approximation was possible due to the time-explicit nature of the numerical procedure. The initial condition of the calculation was the fully developed steady Dean vortex flow at the same Re and pipe curvature. The flow field evolved in time until a constant amplitude traveling wave solution was achieved. No external perturbation was required in the experiment or the calculation to induce the traveling wave.

The streamwise length of the domain for all the calculations was $19\degree$, the value obtained from the flow visualization photographs. Periodic boundary conditions were imposed, thus restricting the traveling wave to a single specified wavelength (or integer divisions of the fundamental wavelength). This restriction is justified since the power spectrum of the experimental velocity time records revealed a single dominant frequency (and a second harmonic). The frequency observed in the time records calculated along the midplane was equal to $1.44$ Hz ($fd/U = 0.38$), as compared to the experimental value of $0.95$ Hz ($fd/U = 0.25$).

### C. Testing the code

The CUTEFLOWS solution procedure has been extensively tested by Treidler, Humphrey et al., Tatsutani et al., Humphrey et al., and Iglesias et al. for calculations in Cartesian and cylindrical coordinate systems. Further testing was conducted to ensure that modifications related to implementing the toroidal coordinate system were correctly accomplished. Two major test cases performed for steady flow regimes are discussed in detail by Webster. The first is the fully developed flow through a curved pipe at $Re = 500$ and $R_c/a = 40$ ($De = 80$). Both two- and three-dimensional calculations of this case yielded steady-state results, in excellent agreement with the steady numerical calculations of Goering. The second test case was the flow developing in a $180\degree$ curved pipe for the experimental conditions of Agrawal et al. This particular test case was for $Re = 2527$ and $R_c/a = 20$ ($De = 565$). Again, the calculated results agreed very well with the measurements.

### D. Comparison to experimental data

Figure 4(a) compares the calculated midplane mean velocity profile with the experimental measurements of Webster and Humphrey. In the outer half of the pipe cross section the experimental values differed from the calculated profile by up to 5%. This discrepancy is of the order of the reported experimental uncertainty. In the inner half of the pipe cross section the agreement between the experimental and numerical profiles is excellent except immediately adjacent to the wall.

Figure 4(b) shows measurements and calculations of the streamwise rms velocity profile on the midplane. The localized increase in the experimental profile near $r/2a = 0.05$ is attributed by Webster and Humphrey to velocity oscillations due to the traveling wave instability. The calculated profile shows a peak of the rms velocity at the same location, confirming that the traveling wave oscillations are responsible for this local increase in the rms. The quantitative difference between the magnitudes of the measured and calculated local peaks is likely due to the fully developed nature of the calculated traveling wave. The experimental data were obtained in the fifth revolution of the coil, which is in the fully developed region for the Dean vortices (see Berger et al.). However, no data are available to indicate how many revolutions are required for a fully developed traveling wave. Given the
quantitative difference between the rms profiles, it appears that the experimental measurements were not in the fully developed regime, in contrast to the computational results (due to the imposed streamwise periodicity). This suggests that the flow is convectively unstable, in agreement with present flow visualization observations and the investigation of Mees22 for a traveling wave instability in a curved duct of square cross section.

V. RESULTS AND DISCUSSION
A. Mean flow results

The cross-stream mean velocity vectors are shown in Fig. 5(a). The figure shows that the cross-stream flow is significantly different from the cartoon in Fig. 1, which is appropriate for low Dean numbers \((De < 100)\). In particular, the shape of the Dean vortices is asymmetric about an imaginary vertical midplane, separating the inner and outer halves of the pipe cross section. The pressure-driven wall layers returning the cross-stream flow to the inner curved wall become thinner as \(De\) increases and are confined to a distance \(0.1a\) from the wall. The cross-stream flow in the core of the pipe directed toward the outer wall is strongest in the region immediately adjacent to the pressure-driven wall layer.

Figure 5(b) shows contours of the streamwise mean velocity, \(\bar{u}_x/U\); and (c) contours of the rms of the velocity.
the largest. A high rms region connecting the two Regions I is also shown.

B. The traveling wave

The traveling wave perturbation to the fully developed Dean vortex flow moves in the streamwise direction at a constant wave speed. There are several frames of reference available to observe the flow. To a stationary observer, the local velocity field oscillates periodically in time as the perturbation travels past. Previous experimental measurements have been in this frame of reference, leading to unsteady time records of velocity. The results in this section are presented at equally spaced planes in the streamwise directions at an instant in time. The variations observed in the streamwise direction are analogous to the temporal variations seen in the experimental data. The objective is to examine the detailed three-dimensional flow field in order to determine the physical mechanism by which the instability maintains itself.

In discussing the imposed boundary condition above, we stated that the flow was perfectly symmetric about the midplane. This is true at every instant and at every streamwise location, thus the instability is a varicose mode. The sequence in Fig. 7 shows the instantaneous velocity field at pipe cross sections spaced \( \frac{1}{8} \lambda \) increments apart. Each cross section in Fig. 7 is divided at the midplane. The upper portion of each cross section shows the cross-stream velocity perturbation [i.e., the cross-stream velocity field minus the cross-stream mean velocity shown in Fig. 5(a)]. The lower portion shows contours of the instantaneous streamwise velocity component.

Comparing the location and shape of the contours of streamwise velocity at each \( \theta \) location verifies that the streamwise velocity varies significantly in Region I. For example, the streamwise velocity in Region I is notably larger at \( \theta = \frac{3\lambda}{4} \) [Fig. 7(e)] than at \( \theta = \frac{5\lambda}{8} \) [Fig. 7(f)]. This is due to the periodic convection, by the cross-stream wall layers, of fluid with relatively high streamwise velocity in the outer half of the pipe cross section into Region I. The perturbation velocity vectors show that the cross-stream flow varies significantly as a function of \( \theta \) both in the core region and in the cross-stream wall layers. The patterns of the cross-stream perturbation velocity vectors are explained by examining the imbalance between the centrifugal force and the cross-stream pressure gradient in Region I. The following discussion will illustrate that the variation of the centrifugal force in Region I corresponds directly to a variation of cross-stream velocity in the cross-stream wall layer, and that the variation of the cross-stream pressure gradient in Region I corresponds directly to a cross-stream velocity variation in the core region.

Figure 8 shows selected cross-section views of the pressure gradient perturbation in the \( x \) direction (\( \frac{dp}{dx} - \frac{d\tilde{p}}{dx} \)) and the centrifugal force perturbation (\( u_\theta \frac{\partial \xi}{\partial x} - \tilde{u}_\theta \frac{\partial \xi}{\partial x} \)). Strictly, the latter quantity has the units of acceleration, not force, since the constant mass has been neglected. The \( x \) and \( y \) coordinate directions are shown in Fig. 1, where the positive \( x \) direction is pointed toward the outer wall. Positive centrifugal force is also directed toward the

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**Figure 6.** Sketch indicating the regions of the flow in the coiled pipe cross section.
outer wall. The cross section is again divided into halves; the top half showing contours of the pressure gradient perturbation and the bottom half showing contours of the centrifugal force perturbation. Other cross-section plots of the flow, not shown here, are available in Webster.9

The following is true at every cross section. A positive perturbation of the centrifugal force in Region I decreases the velocity component directed toward the inner wall in the cross-stream wall layer, while a negative perturbation increases this velocity. A positive perturbation of the pressure gradient in Region I decreases the flow through the core toward the outer wall while a negative perturbation increases this flow.

The following two examples illustrate these correlated effects. Figure 8(a) (at $\theta=\frac{1}{4}\lambda$) shows that the centrifugal force perturbation in Region I is negative and that the corresponding velocity perturbation in the cross-stream wall layer is directed toward the inner wall [Fig. 7(b)]. Physically, this means that a decrease in the centrifugal force in Region I leads to an increase in the velocity of fluid in the cross-stream wall layer directed toward the inner wall. The corresponding pressure gradient perturbation in Region I is negative [Fig. 8(a)] and the velocity perturbation in the core region, although weak, is toward the outer wall [Fig. 7(b)]. Thus, a decrease in the cross-stream pressure gradient increases the cross-stream velocity directed toward the outer wall in the core region.

The converse situation is observed at $\theta=\frac{3}{4}\lambda$. Figure 8(b) shows that the centrifugal force in Region I is larger than average. The corresponding cross-stream velocity perturbation vectors provided in Fig. 7(f) show a flow directed toward the outer wall in the cross-stream wall layer. Hence, the increased centrifugal force in Region I slows down the flow in the cross-stream wall layer directed toward the inner wall. The cross-stream pressure gradient perturbation is positive in Region I [Fig. 8(b)] and the cross-stream velocity perturbation in the core region is directed toward the inner wall [Fig. 7(f)]. The result is for the increased cross-stream pressure gradient to reduce the velocity component directed toward the outer wall in the core region.

![Fig. 7. The cross-stream velocity perturbation vector field ($u_r\tilde{u}_r$, $u_f\tilde{u}_f$) (top) and contours of the streamwise velocity, $u_f/U$ (bottom) at (a) $\theta=0$ and $\lambda$; (b) $\theta=\frac{1}{4}\lambda$; (c) $\theta=\frac{3}{4}\lambda$; (d) $\theta=\frac{5}{4}\lambda$; (e) $\theta=\frac{7}{4}\lambda$; (f) $\theta=\frac{9}{4}\lambda$; (g) $\theta=\frac{11}{4}\lambda$; and (h) $\theta=\frac{13}{4}\lambda$.](image-url)
It is important to understand the variation of the cross-stream velocity in order to describe the physical mechanism for the energy transfer from the bulk streamwise flow to the traveling wave and so explain the mechanism maintaining the instability. As observed in Fig. 7, at any instant in time the magnitude of the streamwise velocity component in Region I varies periodically in the streamwise direction. When the streamwise velocity is relatively large in Region I, the centrifugal force is correspondingly large and directed toward the outer wall. As described above, the velocity component in the cross-stream wall layer directed toward the inner wall is thus reduced and its ability to convect fluid with large streamwise velocity into Region I is correspondingly diminished. The centrifugal force thus decreases, which increases the cross-stream wall layer flow toward the inner wall, which in turn increases the streamwise velocity in Region I. This cycle repeats itself indefinitely since it is energized by the bulk streamwise velocity through the periodic variation in centrifugal force.

C. Discussion of the flow instability

The details of the traveling wave have been examined and the mechanism for maintaining the instability has been described by examining the flow at consecutive cross-sectional planes. However, the cause of the instability is still unclear. By comparing the contours of the rms velocity [Fig. 5(c)] to the mean cross-stream velocity vectors [Fig. 5(a)] it is clear that the region of maximum velocity perturbation is located at the critical point where the flow in the cross-stream wall layer is turned into the core region. The cross-stream flow in this region is subject to a centrifugal force due to the concave curvature of the pipe wall [see the $u^2/r$ term in Eq. (3)]. The centrifugal force in this discussion is differ-
Rayleigh’s circulation criterion states that a necessary and sufficient condition for stability to axisymmetric disturbances is that the quantity $\Phi$ be positive. This means that the square of the circulation must increase in the radial direction in order for the flow to be centrifugally stable. Physically, $\Phi$ is a measure of the ability of the inward-directed radial pressure gradient to correct for an infinitesimal outward-directed radial centrifugal perturbation. If the Rayleigh discriminant is negative, it indicates that the flow has the potential for a centrifugal instability.

Known cases of centrifugal instability, such as the shear-driven flow between corotating cylinders, the pressure-driven flow in a curved duct, and the boundary layer flow along a longitudinally concave wall, all produce flow perturbations consisting of pairs of counter-rotating vortices (respectively, Taylor vortices, Dean vortices, and Görtler vortices). The vectors in Fig. 9(a) show that the perturbation to the flow projected onto the $r$-$\theta$ plane at $\phi=129^\circ$ consists of two counter-rotating vortices centered around $r/\alpha=0.25$, approximately. A relatively strong flow toward the wall is observed between the counter-rotating vortices at approximately $\theta=\pm 0.18$, and a corresponding flow away from the wall is observed around $\theta=0.28$. The $\omega_\phi$ perturbation plot [Fig. 9(b)] shows corresponding regions of opposite-signed vorticity. A sequence of plots for neighboring $r$-$\theta$ planes passing through Region I is provided in Webster. They show the presence of the counter-rotating vortices throughout the region. The unusual shape of the present vortex pair is due to the mean streamwise velocity gradient that is not present in the relatively simple flows listed above.

The Rayleigh discriminant profile shown in Fig. 9(c) is negative near the wall and in an isolated region between $r/\alpha=0.34$ and 0.4. The Rayleigh discriminant is, in fact, negative in the near wall region all along the cross-stream wall layer, while the isolated negative region is a unique attribute for the profiles in Region I (at other $\phi$ locations the velocity gradient is insufficient to produce a negative Rayleigh discriminant away from the wall). The isolated negative region is also only observed at large Dean number because the cross-stream flow at lower $De$ appears like the sketch in Fig. 1 and the Rayleigh discriminant is not negative away from the wall. Hence, for sufficiently large Dean number the Rayleigh discriminant in Region I only becomes negative away from the wall, and the traveling wave instability appears with the strongest perturbation also in Region I. Thus, it is proposed that the cross-stream flow becomes centrifugally unstable and induces the traveling wave instability.

In recent years traveling wave instabilities have been observed in several curved flows. These include the Taylor–Couette flow (Marcus, among others), the flow in a curved channel (Finley et al. and Le Cunff and Bottaro), the flow between corotating disks (Humphrey et al.), and the flow in a curved duct of square cross section (Mees). These geometries are similar among themselves because the traveling wave instability is of the sinuous mode type and the traveling waves are believed to result from a shear instability. In the curved channel, Le Cunff and Bottaro showed that the sinuous mode was always more unstable than the varicose

The difference in the geometry and resulting flow stream flow is subject to the concave curvature of the pipe. Geometrically, the current flow differs because the cross-stream flow is similar to these flow geometries in Fig. 9.

The traveling wave instability visualized in the transitional regime was simulated using a finite-difference based numerical procedure for solving the Navier–Stokes equations formulated in a toroidal coordinate system. The results reveal a physical mechanism explaining the transfer of energy from the base flow to the traveling wave. Understanding the effect of the streamwise variation of the centrifugal force on the cross-stream flow is, indeed, centrifugally unstable requires an instability analysis beyond the scope of this work.

VI. CONCLUSION

Flow visualization photographs of the flow in a helical coil for a range of Reynolds numbers spanning the laminar and turbulent regimes have been presented and discussed. The traveling wave instability visualized in the transitional regime was simulated using a finite-difference based numerical procedure for solving the Navier–Stokes equations formulated in a toroidal coordinate system. The results reveal a physical mechanism explaining the transfer of energy from the base flow to the traveling wave. Understanding the effect of the streamwise variation of the centrifugal force on the cross-stream velocity in a subregion of the flow, Region I, is fundamental to the discussion. An examination of the flow perturbation projected onto the $r-\theta$ plane allows an analogy to be made with known centrifugal instabilities. Simple considerations show that the cross-stream flow has the potential for a centrifugal instability. This instability is quite distinct from the centrifugal force due to the streamwise velocity, which induces the classical Dean vortex pair.

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20. K. Tatsutani, W. R. Usry, and J. A. C. Humphrey, “Numerical solution of...


